On the DeWitt metric

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Abstract. *The DeWitt metric on the space of Riemannian metrics of a compact manifold naturally arises in a Hamiltonian description of General Relativity. We provide a synopsis of the differential-geometric properties of this metric. A potential link to superspace topology is discussed.*

1. NOTATIONS

Let M be a compact smooth manifold of dimension $n \geqslant 2$, $\mathscr{F}(M)$ the algebra of real functions on M, $\chi(M)$ the space of vector fields on M, $\mathscr{S}(M)$ the space of symmetric bilinear forms on M, $\mathscr{M}$ the open cone of Riemannian metrics of M in $\mathscr{S}(M)$ and $\mathscr{D}$ the group of diffeomorphisms of M which acts on $\mathscr{M}$ via the pull-back operation.

For a given $g \in \mathscr{M}$, let ∇ denote the Levi-Civita connection, R the Riemann curvature tensor, ρ (resp. $\check{\rho}$) the Ricci curvature (resp. ρ acting on $\chi(M)$), τ the scalar curvature, ν_g the volume element of g. For $h \in \mathscr{S}(M)$, $g^{-1}(h)$ designates the trace with respect to g. We shall not make explicit the musical isomorphisms between the tangent bundle TM and the cotangent bundle T^*M via g.

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There exists a family $(\,.\,,.\,)^{(\alpha)}$ of local scalar products on $\mathcal{M}$ depending on a real parameter α. Identifying the tangent space of $\mathcal{M}$ at g with $\mathcal{S}(M)$, for $h,\ k \in \mathcal{S}(M)$ one defines

$$(1.1)\qquad (h,\ k)^{(\alpha)} = g^{-2}(h \otimes k) + \alpha g^{-1}(h) g^{-1}(k)$$

The corresponding global scalar product $\langle .,\ .\rangle^{(\alpha)} = \int_M (.,\ .)^{(\alpha)} \nu_g$ is invariant under the action of $\mathcal{D}$, positive definite for $\alpha \geqslant -1/n$ and nondegenerate for $\alpha \neq -1/n$. For the weight $\alpha = 0$ it is the canonical metric, for $\alpha = -1$ it is the DeWitt metric (we shall write symbolically DW instead of -1). The DeWitt metric first arose in a Hamiltonian description of General Relativity [4]. The term itself was coined in [7] and [14]. The corresponding global norms are written as $\|\cdot\|_{(\alpha)}$. These are true norms only for $\alpha > -1/n$ but we shall write $\|h\|^2_{(\alpha)} = (h,h)^{(\alpha)}$ for any α. We shall also need the triple product $(h,\ k,\ p) = g^{-3}(h \otimes k \otimes p)$ of three symmetric bilinear forms. This is unambiguously defined.

Let d stand for the exterior derivative of the De Rham complex of M, $\delta = \delta(g)$ for its formal adjoint with respect to g, δ_g^* for the differential operator from $\chi(M)$ to $\mathcal{S}(M)$ which to a vector field ξ associates $\delta_g^*\xi = \frac{1}{2} L_\xi g$. For $\alpha \neq -1/n$, $\delta_g^{(\alpha)} = \delta_g^{(0)} - \alpha\, \mathrm{d}g^{-1}(\ .\ )$ is the formal adjoint of δ_g^* with respect to $\langle .,.\rangle^{(\alpha)}$. Let Hess: $\mathcal{F}(M) \to \mathcal{S}(M)$ denote the Hessian which to $f \in \mathcal{F}(M)$ associates Hess $f = \nabla\, \mathrm{d}f$. The Laplacian Δ acting on $\mathcal{F}(M)$ is the opposite of the trace of the Hessian.

2. THE DeWITT METRIC AS A CRITICAL METRIC ON $\mathcal{M}$

We may rewrite the fundamental identities of Riemannian geometry as follows:

$$(2.1)\qquad \begin{aligned} &\delta_g^{(\alpha)}\rho = -\left(\alpha + \frac{1}{2}\right)\mathrm{d}\tau && \text{(Bianchi identity)} \\ &\delta_g^{(\alpha)}\delta_g^* = (\alpha+1)\,\mathrm{d}\delta + \frac{1}{2}\,\delta d - \check{\rho} && \text{(Ricci identity)} \end{aligned}$$

The differential operator $\delta_g^{(\alpha)}\delta_g^*$ on $\chi(M)$ is self-adjoint if $\alpha \neq -1/n$, elliptic if $\alpha > -1$ and hyperbolic if $\alpha < -1$. The parabolic case corresponds to the DeWitt metric. For $\alpha > -1$, the Ricci identity in (2.1) compares two Laplacians, $\Delta_g^{(\alpha)} = \delta_g^{(\alpha)}\delta_g^*$ and $(\alpha + 1)\,\mathrm{d}\delta + \frac{1}{2}\,\delta d$, i.e., it is a Weitzenböck formula.

From an extrinsic point of view the DeWitt metric is distinguished by the following property [4]. If $(M,\ g)$ is a Riemannian hypersurface of a Lorentzian manifold with second fundamental form K, then one has $S_L = S_R + \|K\|^2_{(DW)}$ where S_L (resp. S_R) is the total Lorentzian (resp. Riemannian) scalar curvature.

The geodesics of the DeWitt metric were computed in the seminal paper [4] whereas the geodesics of the canonical metric have been determined in [5] and [11].

3. A DECOMPOSITION THEOREM

In [2] Berger and Ebin show that for $\alpha = 0$ the tangent space of $\mathcal{M}$ at g splits as the orthogonal sum with respect to the global inner product $\langle .,. \rangle^{(\alpha)}$:

$$\mathcal{S}(M) = Im\ \delta_g^* \oplus Ker\ \delta_g^{(\alpha)}. \tag{3.1}$$

In fact, the same result holds for any coupling constant $\alpha \neq -1/n$. The image of δ_g^* contains the fundamental or vertical vector fields which are tangent to the orbit of $\mathcal{D}$ at g. The kernel of $\delta_g^{(\alpha)}$ consists of α-horizontal vectors which are tangent to a transversal slice at g [5].

PROPOSITION 3.2. *For $\alpha \neq -1/n$, the most general α-horizontal vector field on $\mathcal{M}$ whose expression contains derivatives of order at most 2 in the Riemannian metric is*

$$g \mapsto c_0 g + \sum_{k=1}^{[\frac{n}{2}]} c_k \left(\mathrm{Ric}_g^{\{k\}} - \frac{2k\alpha + 1}{2k(1+\alpha n)} \mathrm{Scal}_g^{\{k\}} \cdot g \right)$$

where $\mathrm{Ric}_g^{\{k\}} \in \mathcal{S}(M)$ is the generalized Ricci curvature,

$$(\mathrm{Ric}_g^{\{k\}})_{ij} = \delta^{i_1 \dots i_{2k}}_{j j_2 \dots j_{2k}} R_{i_1 i_2 i}{}^{j_2} R_{i_3 i_4}{}^{j_3 j_4} \dots R_{i_{2k-1} i_{2k}}{}^{j_{2k-1} j_{2k}}, \quad 1 \leqslant k \leqslant \left[\frac{n}{2}\right],$$

$\mathrm{Scal}_g^{\{k\}} = g^{-1}(\mathrm{Ric}_g^{\{k\}})$ *is the generalized scalar curvature and the c_l for $0 \leqslant l \leqslant [\frac{n}{2}]$ are constants.*

Proof. For $\alpha = 0$ the result is due to Lovelock [13]. The generalization is achieved by a simple adjustment of constants taking into account the generalized Bianchi identity [13]

$$\delta_g^{(0)} \mathrm{Ric}_g^{\{k\}} = -\frac{1}{2k} d\ \mathrm{Scal}_g^{\{k\}}.$$

■

4. KILLING FIELDS ON $\mathcal{M}$

The manifold $\mathcal{M}$ being an open cone in $\mathcal{S}(M)$, the deformations $g \mapsto g + th$ with $g \in \mathcal{M}$, $h \in \mathcal{S}(M)$ are flows on $\mathcal{M}$ with $|t|$ small enough. Thus the Lie derivative $\mathscr{L}$ on $\mathscr{M}$ can be computed in terms of directional derivatives. A vector field P on $\mathscr{M}$ is a Killing field for the global scalar product $\langle .,.\rangle^{(\alpha)}$ if and only if $\mathscr{L}_P\langle .,.\rangle^{(\alpha)} = 0$.

PROPOSITION 4.1. *Suppose* $\alpha n \neq -2$. *Then the vector space of Killing fields for* $(\mathscr{M}, \langle .,.\rangle^{(\alpha)})$ *consists of fundamental vector fields except in dimension* 4 *where the tautological field* $g \mapsto g$ *is also a generator (even if* $\alpha = -1/2$*).*

Proof. Let H, K, P be generic vector fields on $\mathscr{M}$ with pointwise values $H(g) = h$, $K(g) = k$, $P(g) = p$. If P is to be a Killing field with respect to a coupling constant α one finds by inspection that any H and K should satisfy

$$\int_M \Big(-2(h,k,p) + \frac{1}{2}(h,k)^{(0)}g^{-1}(p) - \alpha(h,p)^{(0)}g^{-1}(k) - \alpha(k,p)^{(0)}g^{-1}(h) + \frac{\alpha}{2}g^{-1}(h)g^{-1}(k)g^{-1}(p)\Big)\nu_g = 0 \tag{4.2}$$

at any point $g \in \mathscr{M}$. Choose in particular $h = k = g$. Then (4.2) reduces to

$$\left(\frac{\alpha}{2}n^2 + \left(\frac{1}{2} - 2\alpha\right)n - 2\right)\int_M g^{-1}(p)\nu_g = 0. \tag{4.3}$$

Suppose for the moment that $\int_M g^{-1}(p)\nu_g \neq 0$. Then the second order equation remaining from (4.3) admits the roots $n = 4$ and $n = -1/\alpha$ (if $\alpha \neq 0$). The latter one corresponds to the degenerate scalar product.

In the case $n = 4$, the special choice $h = g$, $k = p$ in (4.2) implies

$$\int_M (g^{-1}(p))^2\nu_g = 4\,\|p\|^2_{(0)}$$

for any coupling constant $\alpha \neq -1/2$. Hence, P must be colinear with the tautological field. On the other hand, by inspection one sees that if P is the tautological field then for $n = 4$ (4.2) is identically satisfied for any α. Moreover, the tautological field does not satisfy (4.2) for any α if $n \neq 4$.

In the degenerate case $\alpha = -1/n$, the choice $h = g$, $k = p$ in (4.2) yields

$$\int_M (g^{-1}(p))^2\nu_g = n\,\|p\|^2_{(0)}$$

whence P must be colinear with the tautological field and hence $n = 4$.

Next, study the case where $\int_M g^{-1}(p)\nu_g = 0$. This includes the fundamental fields as by Green's theorem

$$(4.4) \qquad \int_M g^{-1}(\delta_g^* \xi) = -\int_M \delta\xi = 0$$

for any $\xi \in \chi(M)$. The fundamental fields are Killing fields for any α as $\langle .,. \rangle^{(\alpha)}$ is $\mathscr{D}$-invariant. The tautological field is not a fundamental field because if it was, we would find $\int_M g^{-1}(g) = n$ volume$(M) = 0$ by (4.4).

If $p \in \mathscr{S}(M)$ satisfies $\int_M g^{-1}(p)\nu_g = 0$ then it can be decomposed as $p = \delta_g^*\xi + \tilde{p}$ with $\xi \in \chi(M)$ and $\tilde{p}$ trace-free. We still need to check whether the trace-free vector fields on $\mathscr{M}$ provide more generators for the vector space of Killing fields. In general, this is not the case as for the choice $h = g$, $k = p = \tilde{p}$ (4.2) reduces to

$$(\alpha n + 2)\, \| \tilde{p} \|^2_{(0)} = 0.$$

However, in the case $\alpha n = -2$ we are not able to conclude. This leaves open the problem of determining the DeWitt Killing fields on the space of Riemannian metrics on a compact Riemann surface. On the space $\mathscr{M}_{-1}$ of Poincaré metrics on a compact Riemann surface of genus > 1 there is no DeWitt metric as deformations of Poincaré metrics are necessarily trace-free [8] and the twisting term in (1.1) drops away. The quotient space of $\mathscr{M}_{-1}$ by the identity component of $\mathscr{D}$, or the classical Teichmüller space, is a finite-dimensional manifold [8]. The canonical metric on $\mathscr{M}$, or as well on $\mathscr{M}_{-1}$, projects down to the Teichmüller space to yield its classical Weil-Petersson metric up to a constant factor [9]. Some results on the Killing fields of this metric can be found in [15]. ■

5. A FORMULA ON THE DeWITT HORIZONTALS

Let ω be an l-form on $\mathscr{M}$ with values in any vector space E. One defines the exterior derivative $d_{\mathscr{M}}$ on $\mathscr{M}$ by setting for arbitrary vector fields $H_0, \ldots, H_l$ on $\mathscr{M}$

$$\mathrm{d}_{\mathscr{M}}\omega(H_0, \ldots, H_l) = \sum_{i=0}^{l} (-1)^i \mathscr{L}_{H_i}(\omega(H_0, \ldots, \hat{H}_i, \ldots, H_l))$$

$$+ \sum_{i<j} (-1)^{i+j} \omega([H_i, H_j], H_0, \ldots, \hat{H}_i, \ldots, \hat{H}_j, \ldots, H_l)$$

where $[H_i, H_j] = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \{H_i(g + tH_j(g)) - H_j(g + tH_i(g))\}$.

The scalar curvature τ is a 0-form on $\mathcal{M}$ with values in $\mathcal{F}(M)$ whereas the Ricci curvature ρ is a vector field on $\mathcal{M}$. The vector fields $g \mapsto \rho + \beta\tau \cdot g$ where β is a real parameter give rise to a family of 1-forms $\rho + \beta\tau \cdot g^{(\alpha)}$ on $\mathcal{M}$ with values in $\mathcal{F}(M)$ which are the dual 1-forms of $\rho + \beta\tau \cdot g$ with respect to each local scalar product $(.,.)^{(\alpha)}$.

In this setting the classical first variation formula of the scalar curvature [3] takes the following form

$$d_{\mathcal{M}}\tau = -\bar{\rho}^{(0)} + \delta \circ \delta_g^{(DW)}. \tag{5.1}$$

PROPOSITION 5.2. *On the DeWitt horizontals*

$$d_{\mathcal{M}}\left(\overline{\rho - \frac{1}{n-1}\tau \cdot g}^{(DW)}\right) = d_{\mathcal{M}}\,\bar{\rho}^{(0)} = \delta \circ d_{\mathcal{M}}\,\delta_g^{(DW)}. \tag{5.3}$$

Proof. The first equality in (5.3) is trivial by inspection. Applying $d_{\mathcal{M}}$ to both sides of (5.1), one finds

$$d_{\mathcal{M}}\,\bar{\rho}^{(0)} = d_{\mathcal{M}}(\delta \circ \delta_g^{(DW)}).$$

Suppose that two arbitrary tangent vectors $h, k \in T_g\mathcal{M}$ are extended to constant vector fields in a neighbourhood of g. Now both $\delta = \delta(g)$ and $\delta_g^{(DW)}$ are linear differential operators on fixed spaces for a fixed g so the Leibniz rule applies and we may compute

$$\begin{aligned}
&d_{\mathcal{M}}(\delta \circ \delta_g^{(DW)})(h,k)\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(\delta(g+th)\circ\delta_{g+th}^{(DW)})(k) - \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(\delta(g+tk)\circ\delta_{g+tk}^{(DW)})(h)\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\delta(g+th)\circ\delta_g^{(DW)}(k) - \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\delta(g+tk)\circ\delta_g^{(DW)}(h)\\
&\quad + \delta(g)\circ\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\delta_{g+th}^{(DW)}(k) - \delta(g)\circ\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\delta_{g+tk}^{(DW)}(h).
\end{aligned}$$

In the kernel of $\delta_g^{(DW)}$ the first two terms of the last expression above vanish and the sum of the remaining ones equals $\delta \circ d_{\mathcal{M}}\,\delta_g^{(DW)}$. This proves the second equality in (5.3). ∎

One might conjecture that the second equation of (5.3) can be generalized for other coupling constants α on the right hand side if some suitable counter terms are added to the 1-form $\bar{\rho}^{(0)}$. In fact, this only works for $\alpha = -\frac{1}{2}$. Besides the DeWitt metric, this particular choice of the coupling constant also has some interest but we prefer to postpone its discussion to an appendix.

6. PERSPECTIVES

In this section we comment on Proposition 5.2 which we view as the main result of this paper. The action of $\mathscr{D}$ on $\mathscr{M}$ is not free in general. However, the singularities in the «superspace» $\mathscr{M}/\mathscr{D}$ can be unfolded in several ways [6]. For instance, it is enough to reduce $\mathscr{D}$ to the subgroup $\mathscr{D}'$ of the diffeomorphisms that «strongly fix a point», i.e., which fix a base point of M and a frame at it. The associated fibration over $\mathscr{M}/\mathscr{D}'$ is a principal fibre bundle which may not admit a global gauge [12].

The decomposition (3.1) is easily seen to be equivariant with respect to the action of $\mathscr{D}$ in the sense that Ker $\phi^* \delta_g^{(\alpha)} = \phi^*$ Ker $\delta_g^{(\alpha)}$ for $\phi \in \mathscr{D}$. Hence, the horizontal distributions $g \mapsto$ Ker $\delta_g^{(\alpha)}$ give a family of connections of the bundle $\mathscr{M} \to \mathscr{M}/\mathscr{D}'$. The associated connection 1-forms formally read $(\delta_g^{(\alpha)} \delta_g^*)^{-1} \delta_g^{(\alpha)}$. For $\alpha \geqslant 0$, the kernel of the Laplacian $\Delta_g^{(\alpha)} = \delta_g^{(\alpha)} \delta_g^*$ consists of the infinitesimal isometries of (M, g). In the absence of these, the Green's operator $(\Delta_g^{(\alpha)})^{-1}$ does make sense. The associated curvature 2-form on the α-horizontals formally reads $d_{\mathscr{M}} ((\Delta_g^{(\alpha)})^{-1} \delta_g^{(\alpha)})$.

We want to point out the tantalizing resemblance that the term $d_{\mathscr{M}}\, \delta_g^{(DW)}$ of (5.3) bears to the ill-defined curvature 2-form coming from the DeWitt metric. On the other hand, we may view the divergence δ (up to a constant factor) as the «infinite-dimensional trace» on $\chi(M)$. Indeed, the volume element ν_g is the good infinite-dimensional analogue of the finite-dimensional determinant and the well-known formula $\mathscr{L}_\xi \nu_g = -(\delta\xi)\nu_g$ should be viewed as the infinite-dimensional version of the usual derivation rule of the determinant. Proposition 5.2 thus might reflect the existence of a transgression formula for the divergence of the suitably regularized curvature (an «infinite-dimensional first Chern class») of the bundle $\mathscr{M} \to \mathscr{M}/\mathscr{D}'$. Analogous developments appear in recent literature [1], [10] but we have not been able to relate them in a rigorous way with formula (5.3).

APPENDIX: A TRANSGRESSION FORMULA FOR $\alpha = -\frac{1}{2}$

Let ω denote the 1-form on $\mathscr{M}$ with values in $\mathscr{F}(M)$ which to $h \in \mathscr{S}(M)$ associates

$$\omega_g(h) = (\rho, h)^{(0)} - \frac{1}{2}\Delta(g^{-1}(h)). \tag{A1}$$

Then the following generalizes the second equality of (5.3):

PROPOSITION. *In the kernel of* $\delta_g^{(-1/2)}$, *the following holds*

$$d_{\mathcal{M}}\,\omega = \delta \circ d_{\mathcal{M}}\,\delta_g^{(-1/2)}. \tag{A2}$$

The discussion on the meaning of formula (5.3) given in section 6 applies verbatim to (A2) also. In particular, exactly the same difficulties as before arise with the ill-defined Green's operator $(\delta_g^{(-1/2)}\delta_g^*)^{-1}$. Thus the transgression formula (A2) is no better than (5.3) but we find it worthwhile to put it on record as formulas of this type are certainly hard to come by.

We now sketch the proof of (A2). First of all, the reader can check by a tedious but straightforward computation, say, in normal coordinates that for $h, k \in \operatorname{Ker} \delta_g^{(\alpha)}$ and for any α one has on the one hand

$$\begin{aligned} \delta \circ d_{\mathcal{M}}\,\delta_g^{(\alpha)}(h, k) &= \frac{1}{2}(\Delta h, k)^{(0)} - \frac{1}{2}(\Delta k, h)^{(0)} \\ &+ \frac{1}{2}(\operatorname{Hess}\operatorname{Tr} h, k)^{(0)} - \frac{1}{2}(\operatorname{Hess}\operatorname{Tr} k, h)^{(0)} \quad , \end{aligned} \tag{A3}$$

and on the other hand

$$\begin{aligned} d_{\mathcal{M}}\,\bar{\rho}^{(0)}(h, k) &= \frac{1}{2}(\Delta h, k)^{(0)} - \frac{1}{2}(\Delta k, h)^{(0)} \\ &- \left(\alpha + \frac{1}{2}\right)((\operatorname{Hess}\operatorname{Tr} h, k)^{(0)} - (\operatorname{Hess}\operatorname{Tr} k, h)^{(0)}). \end{aligned} \tag{A4}$$

Comparing (A3) and (A4) with $\alpha = -1$ gives another proof of the second equality of (5.3).

Moreover, for $\alpha = -\frac{1}{2}$, (A4) simplifies to

$$d_{\mathcal{M}}\,\bar{\rho}^{(0)}(h, k) = \frac{1}{2}(\Delta h, k)^{(0)} - \frac{1}{2}(\Delta k, h)^{(0)}. \tag{A5}$$

On the other hand, by a variation formula given in [3] one has

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Delta_{g+th} f = (\operatorname{Hess} f, h)^{(0)} - g(\mathrm{d}f, \delta_g^{(-1/2)}h). \tag{A6}$$

For $h \in \operatorname{Ker} \delta_g^{(-1/2)}$, the second term on the right hand side of (A6) drops away. This inspires us to seek the suitable correction term to be added to $\bar{\rho}^{(0)}$ in the form $c\, \Delta(g^{-1}(.))$ for some constant c. Indeed, one readily computes that for h, $k \in \operatorname{Ker} \delta_g^{(-1/2)}$

(A7) $$d_{\mathscr{M}}\,(\Delta(g^{-1}(.)))(h, k) = (\text{Hess Tr } k, h)^{(0)} - (\text{Hess Tr } h, k)^{(0)}.$$

Hence, adding $c = -\frac{1}{2}$ times the right hand side terms of (A7) to those of (A5) one exactly recovers the right hand side terms of (A3). This proves (A2). ∎

Let us finally point out that by (2.1) the Ricci curvature vector field is horizontal for $\alpha = -1/2$.

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